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Equations for the extension and flexure of electroelastic plates under strong electric fields

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Abstract

A system of approximate equations for the extensional and flexural motion of electroelastic plates subject to large electric fields is derived from the variational equations of electroelasticity for small strain and cubic electric fields. The two-dimensional equations are derived by introducing an appropriate expansion for the mechanical displacement and electric potential in the thickness-coordinate and integrating through the thickness. The resulting equations are reduced to the uncoupled systems of anisotropic extension and elementary flexure. The electroelastic constitutive equations are reduced to the form suitable for thin plates by relaxing the appropriate plate stress resultants. © 1999 Elsevier Science Ltd. All rights reserved.

1. Introduction

Many piezoelectric components have plate-like shapes. Some of these components like resonators and filters are for the purpose of frequency control. They are often made of materials such as quartz with weak piezoelectric coupling and work under small electric fields. Several versions of linear piezoelectric plate theories for these applications have been derived by Tiersten and Mindlin (1962), Mindlin (1972), and Lee et al. (1987). Piezoelectric components like actuators and transformers are used to transfer force or power. They usually work under relatively strong electric fields and are made of materials with strong piezoelectric coupling like polarized ceramics or lithium niobate. For these applications nonlinearity due to strong electric fields often needs to be considered. For example, the saturation of the output voltage of a piezoelectric transformer is a consequence of electric nonlinearity. The need to consider electric nonlinearity in smart structures has also been noted by Rao and Sunar (1994). Particularly, the important phenomenon of electrostriction and the related biased piezoelectricity is based on quadratic electric nonlinearity.

Recently electroelastic equations for the extensional motion of very thin electroelastic plates

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with fully electroded major surfaces subject to large driving voltages and undergoing small strain were derived by Tiersten (1993a) from the general nonlinear electroelastic equations of Baumhauer and Tiersten (1973) and Tiersten (1975). A system of approximate two-dimensional equations were subsequently obtained by Tiersten (1993b) for a higher-order description for the mechanical and quadratic nonlinear electrical behavior of relatively thin electroelastic plates in extensional and flexural motion, and the equations are valid for either electroded or unelectroded plates. These equations were later generalized to thermoelectroelastic plates to include thermal fields by Yang and Batra (1994, 1995a). The equations of Tiersten (1993a, b) have been used by Tiersten (1993c), Zhou and Tiersten (1994), Yang et al. (1994), Batra et al. (1996a, b), Yang and Tiersten (1997), Yang (1997), and Yang (1998) to analyze various problems of elastic plates with fully or partially electroded piezoelectric actuators and buckling of piezoelectric plates.

In this paper, the plate equations of Tiersten (1993b) are generalized to include cubic electrical nonlinearity. This is because very often quadratic nonlinearity does not give enough accuracy according to Tiersten (1993a) or correct qualitative behavior and sometimes, because of the symmetry of the material, quadratic nonlinear terms do not appear and the first nonlinear terms are cubic terms. The equations derived in this paper also generalize those of Tiersten (1993b) in that the equations in this paper include the lowest order term of the electric potential which is uniform in the plate thickness direction while those of Tiersten (1993b) do not have this lowest order electric potential term. For ceramics poled in the thickness direction this lowest order electric potential term vanishes in certain problems but for the more general case this term has a contribution and it is included in the present work for completeness and generality.

The plate equations in this paper which contain terms up to cubic in the electric field and linear in the mechanical variables, are obtained by means of an appropriate expansion of the mechanical displacement and electric potential in powers of the thickness coordinate in the variational equation of electroelasticity and an integration through the thickness in the manner of Mindlin (1955). Since only the lowest frequency approximations are of interest here, the appropriate assumptions are made and the equations of the extensional and elementary flexural motion of thin plates are obtained, and in this lowest order approximations are uncoupled in the general anisotropic case. The expansion of the electric potential contains terms up to cubic in the thickness coordinate with four independent plate potentials which are defined so that all but the constant and linear ones vanish at the surface of the plate. Because of this, in the electroded region with a prescribed voltage there are only two electrostatic plate equations, while in the unelectroded region there are four electrostatic plate equations.

2. Basic equations

From the fully nonlinear equations of electroelasticity given by Baumhauer and Tiersten (1973) and Tiersten (1975) the electroelastic equations for infinitesimal strain and including terms up to cubic in the electric field may be written in the form

$$\tau_{ij,i} = \rho \ddot{u}_j, \quad D_{i,i} = 0 \tag{2.1}$$

$$\tau_{ij} = c_{ijkl}S_{kl} - e_{kij}E_k - \tfrac{1}{2}\hat{b}_{klij}E_kE_l - \tfrac{1}{6}d_{klmij}E_kE_lE_m$$
$$D_i = e_{ikl}S_{kl} + \varepsilon_{ik}E_k + \tfrac{1}{2}\chi_{kji}E_kE_j + \tfrac{1}{6}\tilde{\chi}_{kjli}E_kE_jE_l \tag{2.2}$$

where

$$S_{kl} = \tfrac{1}{2}(u_{k,l} + u_{l,k}), \quad E_k = -\phi_{,k} \tag{2.3}$$

and we have introduced the conventions that a comma followed by an index l denotes partial differentiation with respect to the referential coordinate x_l, a dot over a variable denotes partial differentiation with respect to time and repeated indices are to be summed. The range of the indices i, j, k, l are 1, 2, 3. In eqns (2.1)–(2.3) ρ, u_i, τ_{ij}, D_i, S_{kl}, E_k and ϕ denote the reference mass density, the mechanical displacement vector, the total stress tensor (mechanical plus Maxwell electrostatic), the electric displacement vector, the small strain tensor, the electric field vector and the electric potential, respectively. Note that the linear portions of the nonlinear constitutive equations in (2.2) are identical with the linear piezoelectric constitutive equations, in which the constants c_{ijkl}, e_{ikl} and ε_{ik} denote the elastic, piezoelectric and dielectric constants, respectively. $\hat{b}_{klij}$, χ_{kji}, d_{klmij} and $\tilde{\chi}_{kjli}$ are nonlinear material constants. We note that $\hat{b}_{klij}$, the effective electrostrictive constant, includes the effect of the Maxwell electrostatic stress tensor.

Since we are interested in obtaining the equations of the extensional and flexural motion of relatively thin electroelastic plates along with the electrostatic plate equations, all of which are two-dimensional, we have recourse to the variational equations of electroelasticity which is consistent with the foregoing system of equations. It is clear from the assumption of infinitesimal strain in the general rotationally invariant electroelastic equations, the form of the equations presented in this section and the variational equation of linear piezoelectricity of Tiersten (1969), that the variational equation of electroelasticity for the approximation employed in this work can be written in the form of Tiersten (1993b)

$$\int_{t_0}^{t} \mathrm{d}t \int_V [(\tau_{lk,l} - \rho\ddot{u}_k)\delta u_k + D_{l,l}\delta\phi]\,\mathrm{d}V = 0 \tag{2.4}$$

Equations (2.4), with the aid of eqn (2.3), will be used to obtain approximate two-dimensional equations of mechanical motion and electrostatics for thin plates in the next section.

3. Two-dimensional equations of balance

A plan view and cross-section of a thin plate with thickness $2h$ are shown in Fig. 1 along with the coordinate system. The top and bottom surfaces are either identically electroded or not. The electrodes in a given region are assumed to have a potential difference or voltage across them.

We now obtain approximate two-dimensional mechanical equations of motion and charge equations of electrostatics for the thin plate shown in Fig. 1 by employing assumed expansion of u_j and ϕ in the thickness coordinate in the variational equations of electroelasticity (2.4) and integrating through the thickness. To this end we write the variational equation of electroelasticity given in (2.4) in a form suitable for obtaining the approximate plate equations in accordance with Fig. 1, thus,

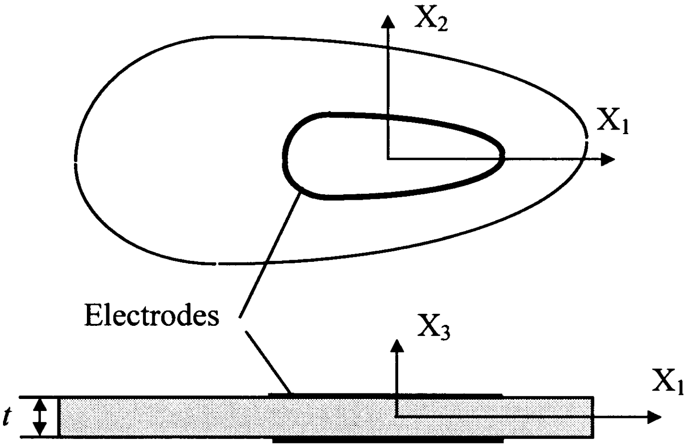

Fig. 1. Plan view and cross-section of a partially electroded plate.

$$\int_S \mathrm{d}S \int_{-h}^{h} [(\tau_{ab,a}+\tau_{3b,3}-\rho\ddot{u}_b)\delta u_b+(\tau_{a3,a}+\tau_{33,3}-\rho\ddot{u}_3)\delta u_3+(D_{a,a}+D_{3,3})\delta\phi]\,\mathrm{d}x_3 = 0 \tag{3.1}$$

in which we have omitted the integral over time, which is unimportant for our purposes, and introduced the convention that a, b, c, d takes the values 1 and 2, but not 3.

Since, as noted in the Introduction, only the lowest frequency approximations, i.e., the equations of the extensional and flexural motion of plates are of interest here, we expand u_j in the form

$$u_j = u_j^{(0)}(x_a,t)+x_3u_j^{(1)}(x_a,t)+x_3^2u_j^{(2)}(x_a,t) \tag{3.2}$$

in which $u_j^{(2)}$ is included to allow for the free thickness-strains accompanying elementary flexure in the manner of Mindlin (1955). Since in an electroded region of the plate the potential difference, or voltage, across the electrode is independent of position in the plane of the plate, we expand ϕ in the form

$$\phi = \phi^{(0)}(x_a,t)+\frac{x_3}{2h}\phi^{(1)}(x_a,t)+\left(\frac{x_3^2}{h^2}-1\right)\phi^{(2)}(x_a,t)+\frac{x_3}{h}\left(\frac{x_3^2}{h^2}-1\right)\phi^{(3)}(x_a,t) \tag{3.3}$$

in which only the second term containing $\phi^{(1)}$ contributes to the voltage across the electrodes. Equation (3.3) is basically taken from Tiersten (1993b), with the difference that here we have $\phi^{(0)}$ but the corresponding equation of Tiersten (1993b) does not and starts with the $\phi^{(1)}$ term. $\phi^{(0)}$ is needed for a complete description in the unelectroded region. In an electroded region the two-dimensional plate electric potentials $\phi^{(0)}$ and $\phi^{(1)}$ must be independent of the coordinates x_a in the plane of the plate. However, in an unelectroded region $\phi^{(0)}$ and $\phi^{(1)}$ are, in general, functions of the coordinates x_a. Clearly, this expansion could be carried to higher-order, but this is the lowest order that is purposeful.

Substituting from (3.2) and (3.3) into (3.1), performing the integrations with respect to x_3, including integrations by parts of the terms differentiated with respect to x_3, we obtain

$$\int_S \left[\sum_{n=0}^{2} \left(\tau_{ak,a}^{(n)} - n\tau_{3k}^{(n-1)} + F_k^{(n)} - \rho \sum_{m=0}^{2} H_{mn}\ddot{u}_k^{(m)} \right) \delta u_k^{(n)} + (D_{a,a}^{(0)} + 2hd_3^{(0)})\delta\phi^{(0)} \right.$$
$$+ \left(\frac{1}{2h} D_{a,a}^{(1)} - \frac{1}{2h} D_3^{(0)} + d_3^{(1)} \right) \delta\phi^{(1)} + \left(\frac{1}{h^2} D_{a,a}^{(2)} - D_{a,a}^{(0)} - \frac{2}{h^2} D_3^{(1)} \right) \delta\phi^{(2)}$$
$$\left. + \left(\frac{1}{h^3} D_{a,a}^{(3)} - \frac{1}{h} D_{a,a}^{(1)} - \frac{3}{h^3} D_3^{(2)} + \frac{1}{h} D_3^{(0)} \right) \delta\phi^{(3)} \right] \mathrm{d}S = 0 \tag{3.4}$$

where

$$\tau_{kl}^{(n)} = \int_{-h}^{h} x_3^n \tau_{kl}\, \mathrm{d}x_3, \quad D_k^{(n)} = \int_{-h}^{h} x_3^n D_k\, \mathrm{d}x_3 \tag{3.5}$$

$$F_j^{(n)} = [x_3^n \tau_{3j}]_{-h}^{h}, \quad d_3^{(n)} = \frac{1}{2h}[x_3^n D_3]_{-h}^{h} \tag{3.6}$$

$$H_{00} = 2h, \quad H_{01} = H_{10} = 0, \quad H_{11} = \tfrac{2}{3}h^3$$
$$H_{02} = H_{20} = \tfrac{2}{3}h^3, \quad H_{12} = H_{21} = 0, \quad H_{22} = \tfrac{2}{5}h^5 \tag{3.7}$$

and we note that the terms $F_j^{(n)}$ and $d_3^{(n)}$ arise from the integrations by parts. We further note that the terms $d_3^{(2)}$ and $d_3^{(3)}$ vanish because the x_3-dependence of the coefficients of $\phi^{(2)}$ and $\phi^{(3)}$ vanishes at $x_3 = \pm h$. For arbitrary δu_k we obtain the nine plate stress resultant equations of motion

$$\tau_{ak,a}^{(n)} - n\tau_{3k}^{(n-1)} + F_k^{(n)} = \rho \sum_{m=0}^{2} H_{mn}\ddot{u}_k^{(m)}, \quad n = 0, 1, 2 \tag{3.8}$$

We note that the three equations for $n = 2$ will not actually occur because they will be eliminated by allowing for the free development of the $u_k^{(2)}$ accompanying anisotropic flexure, as did Mindlin (1955). For arbitrary $\delta\phi^{(n)}$ we obtain the four two-dimensional electric charge equations of electrostatics

$$D_{a,a}^{(0)} + 2hd_3^{(0)} = 0$$
$$\frac{1}{2h} D_{a,a}^{(1)} - \frac{1}{2h} D_3^{(0)} + d_3^{(1)} = 0$$
$$\frac{1}{h^2} D_{a,a}^{(2)} - D_{a,a}^{(0)} - \frac{2}{h^2} D_3^{(1)} = 0$$
$$\frac{1}{h^3} D_{a,a}^{(3)} - \frac{1}{h} D_{a,a}^{(1)} - \frac{3}{h^3} D_3^{(2)} + \frac{1}{h} D_3^{(0)} = 0 \tag{3.9}$$

In an electroded region in which $\phi^{(1)} = V$ (the voltage) $\phi^{(0)}$ is also a constant and the first two equations of (3.9) are not needed to obtain the solution. They simply serve to define $d_3^{(0)}$ and $d_3^{(1)}$ in terms of the plate solution. However, in an unelectroded region in which V is not prescribed these equations are required to obtain a solution.

At this point we have the nine two-dimensional stress equations in (3.8). However, since we are interested in obtaining only the uncoupled equations of anisotropic extension and elementary flexure, it is convenient to rewrite the equations as separate essentially extensional and essentially flexural equations. However, we note that before the elementary uncoupling assumptions are made, the equations are actually coupled due to anisotropy. We further note that we cannot complete the reduction without the plate constitutive relations, which are obtained in the next section. For the plate constitutive equations it is convenient to define Mindlin's plate strains, which we can do in this section simply by substituting the expansion (3.2) into the strain–displacement relation, eqn (2.3), and rearranging terms, which enables us to write

$$S_{ij} = \sum_{n=0}^{2} x_3^n S_{ij}^{(n)}, \quad S_{ij}^{(n)} = \frac{1}{2}[u_{i,j}^{(n)} + u_{j,i}^{(n)} + (n+1)(\delta_{3i}u_j^{(n+1)} + \delta_{3j}u_i^{(n+1)})] \tag{3.10}$$

We now note that we will not need $S_{ij}^{(2)}$ because of the reduction that is to be made. When elementary flexure and extension are to be uncoupled, from (3.8), we may separate the essentially extensional plate equations, which are

$$\begin{aligned}
&\tau_{ab,a}^{(0)} + F_b^{(0)} = 2\rho h\ddot{u}_b^{(0)} + \tfrac{2}{3}\rho h^3 \ddot{u}_b^{(2)} \\
&\tau_{a3,a}^{(1)} - \tau_{33}^{(0)} + F_3^{(1)} = \tfrac{2}{3}\rho h^3 \ddot{u}_3^{(1)} \\
&\tau_{ab,a}^{(2)} - 2\tau_{3b}^{(1)} + F_b^{(2)} = \tfrac{2}{3}\rho h^3 \ddot{u}_b^{(0)} + \tfrac{2}{5}\rho h^5 \ddot{u}_b^{(2)}
\end{aligned} \tag{3.11}$$

and the essentially flexural plate equations, which are

$$\begin{aligned}
&\tau_{a3,a}^{(0)} + F_3^{(0)} = 2\rho h\ddot{u}_3^{(0)} + \tfrac{2}{3}\rho h^3 \ddot{u}_3^{(2)} \\
&\tau_{ab,a}^{(1)} - \tau_{3b}^{(0)} + F_b^{(1)} = \tfrac{2}{3}\rho h^3 \ddot{u}_b^{(1)} \\
&\tau_{a3,a}^{(2)} - 2\tau_{33}^{(1)} + F_3^{(2)} = \tfrac{2}{3}\rho h^3 \ddot{u}_3^{(0)} + \tfrac{2}{5}\rho h^5 \ddot{u}_3^{(2)}
\end{aligned} \tag{3.12}$$

In the case of extension we must first allow for the free plate thickness-strains $S_{33}^{(0)}$ by setting $\tau_{33}^{(0)} = 0$. Then in order to eliminate the first-order extensional equations completely we set $\tau_{3a}^{(1)} = 0$. This has the effect of eliminating the second-order equations in (3.11) completely since all second-order plate stress resultants $\tau_{ij}^{(2)}$ are irrelevant in the approximation, and may be ignored, as may the dynamic terms on the right-hand side of $(3.11)_3$ and $(3.12)_3$. Furthermore, in order to eliminate flexure from the extensional equations, from (3.12) it is clear that we must have $\tau_{a3}^{(0)} = 0$, $\tau_{ab}^{(1)} = 0$, $\tau_{33}^{(1)} = 0$. Collecting all the conditions on the stress resultants, we have

$$\tau_{3j}^{(0)} = 0, \quad \tau_{ij}^{(1)} = 0 \tag{3.13}$$

the first of which will be used to reduce the general plate electroelastic constitutive equations to those that are suitable for anisotropic extension in the next section. We further note that since we have eliminated flexure, we have $\ddot{u}_3^{(0)} = 0$. Also, since we are well below the lowest thickness resonant frequency of the plate, we may assume $\ddot{u}_j^{(1)} = 0$, $\ddot{u}_j^{(2)} = 0$, then all that remain of (3.11) are

$$\tau^{(0)}_{ab,a} + F^{(0)}_b = 2\rho h \ddot{u}^{(0)}_b \tag{3.14}$$

which are the two equations for the extensional motion of thin plates.

In the case of flexure, we must first allow for the plate thickness strain $S^{(1)}_{33}$ by setting $\tau^{(1)}_{33} = 0$. This has the additional effect of eliminating the second-order flexural equation completely because, as already noted, all second-order plate stress resultants are irrelevant in this approximation, and may be ignored. Then, in order to eliminate extension from the flexural equations, from (3.11) it is clear that we must have $\tau^{(0)}_{ab} = 0$, $\tau^{(1)}_{a3} = 0$, $\tau^{(0)}_{33} = 0$. These conditions enable us to write

$$\tau^{(1)}_{3j} = 0, \quad \tau^{(0)}_{ab} = 0, \quad \tau^{(0)}_{33} = 0 \tag{3.15}$$

the first of which will be used to reduce the general plate electroelastic constitutive equations to those that are suitable for elementary flexure of thin plates in the next section. We further note that since we have eliminated extension, we have $\ddot{u}^{(0)}_b = 0$. Also, since we are well below the lowest thickness resonant frequency of the plate, we may assume $\ddot{u}^{(1)}_j = 0$, $\ddot{u}^{(2)}_j = 0$. In order to complete the reduction to the elementary theory of flexure we must take the thickness-shear plate strains $S^{(0)}_{3a}$ to vanish as Mindlin (1955), with which $(3.10)_2$ yields

$$u^{(1)}_a = -u^{(0)}_{3,a} \tag{3.16}$$

which enables us to obtain a single equation in the one dependent variable $u^{(0)}_3$ in the elementary theory of the flexure of thin plates. Utilizing the fact that $\ddot{u}^{(1)}_b = 0$, which is the elimination of rotatory inertia, enables us to obtain $\tau^{(0)}_{3a}$ from $(3.12)_2$, the substitution of which in $(3.12)_1$ yields

$$\tau^{(1)}_{ab,ab} + F^{(1)}_{b,b} + F^{(0)}_3 = 2\rho h \ddot{u}^{(0)}_3 \tag{3.17}$$

which is the equation for the elementary theory of the flexural motion of thin plates.

Thus, at this stage we have the extensional equation of motion (3.14) for which we must obtain plate constitutive equations for $\tau^{(0)}_{ab}$, the flexural equations of motion (3.17), for which we must obtain plate constitutive equations for $\tau^{(1)}_{ab}$, and the four plate equations for electrostatics for which we must obtain plate constitutive equations $D^{(n)}_k$ $(n = 0, 1, 2, 3)$. As noted earlier, the constitutive equations will be obtained in the next section. The constitutive equations will contain the seven dependent variables $u^{(0)}_j$ and $\phi^{(n)}$ $(n = 0, 1, 2, 3)$ as required by the seven field equations (3.9), (3.14) and (3.17).

To the foregoing plate equations we must adjoin the appropriate boundary conditions. For the equations of anisotropic extension (3.14), at an interface separating one region from another we have the well-known continuity conditions

$$n_a [\![\tau^{(0)}_{ab}]\!] = 0, \quad [\![u^{(0)}_b]\!] = 0 \tag{3.18}$$

where n_a denotes the unit normal directed from the '$-$' to the '$+$' side of the interface and, in which we have introduced the usual convention $[\![A]\!] = A^+ - A^-$ for the jump. At the edge either $n_a \tau^{(0)}_{ab}$ or $u^{(0)}_b$ or some combination thereof are prescribed. For the equation of elementary flexure, at an interface we have the well-known continuity conditions

$$n_a [\![\tau^{(1)}_{ab}]\!] n_b = 0, \quad \left[\!\!\left[\tau^{(0)}_{n3} + \frac{\partial \tau^{(1)}_{ns}}{\partial s} \right]\!\!\right] = 0, \quad \left[\!\!\left[\frac{\partial u^{(0)}_3}{\partial n} \right]\!\!\right] = 0, \quad [\![u^{(0)}_3]\!] = 0 \tag{3.19}$$

where

$$\tau_{n3}^{(0)} = n_b \tau_{b3}^{(0)}, \quad \tau_{ns}^{(1)} = n_a \tau_{ab}^{(1)} s_b \tag{3.20}$$

and s_b denotes a unit vector tangent to an interface of separation in the counterclockwise direction. At an edge either $n_a\tau_{ab}^{(1)}n_b$ and $\tau_{n3}^{(0)}+\partial\tau_{ns}^{(1)}/\partial s$ or $\partial u_3^{(0)}/\partial n$ and $u_3^{(0)}$ or some combination thereof are prescribed. In addition, there are the well-known conditions across corners of discontinuous curves.

In the case of the two-dimensional plate equations of electrostatics (3.9), at an interface separating one region from another, we require the continuity conditions

$$[\![\phi^{(n)}]\!] = 0, \quad n = 0, 1, 2, 3 \tag{3.21}$$

since the $\phi^{(n)}$ $(n = 0, 1, 2, 3)$ are constraint type variables. In addition, by multiplying the well-known three-dimensional electrostatic continuity conditions $[\![D_a n_a]\!] = 0$ by $\{1, x_3/2h, x_3^2/h^2-1, (x_3^2/h^2-1)x_3/h\}$, respectively, and integrating the resulting equations from $-h$ to h, we obtain the following two-dimensional plate continuity conditions

$$n_a[\![D_a^{(0)}]\!] = 0, \quad n_a[\![D_a^{(1)}]\!] = 0, \quad n_a[\![D_a^{(2)}-h^2D_a^{(0)}]\!] = 0, \quad n_a[\![D_a^{(3)}-h^2D_a^{(1)}]\!] = 0 \tag{3.22}$$

At an edge either $n_aD_a^{(0)}$, $n_aD_a^{(1)}$, $n_a(D_a^{(2)}-h^2D_a^{(0)})$ and $n_a(D_a^{(3)}-h^2D_a^{(1)})$ or $\phi^{(0)}$, $\phi^{(1)}$, $\phi^{(2)}$ and $\phi^{(3)}$ or some combination thereof are prescribed.

4. Two-dimensional constitutive equations

In this section we obtain the constitutive relations for the plate stress, electric displacement resultants defined in (3.5). The resulting constitutive equations are then reduced to those appropriate for the uncoupled equations of anisotropic extension and elementary flexure by employing eqns $(3.13)_1$ and $(3.15)_1$, respectively. Clearly, the constitutive equations are obtained by substituting from (2.2) into (3.5), employing $(3.10)_1$ and the analogous electrical equations and integrating through the thickness. From $(2.3)_2$ and (3.3) we find that the analogous electrical equations take the form

$$\begin{aligned} E_a &= -\phi_{,a}^{(0)} - \frac{x_3}{2h}\phi_{,a}^{(1)} - \left(\frac{x_3^2}{h^2}-1\right)\phi_{,a}^{(2)} - \frac{x_3}{h}\left(\frac{x_3^2}{h^2}-1\right)\phi_{,a}^{(3)} \\ E_3 &= -\frac{1}{2h}\phi^{(1)} - \frac{2x_3}{h^2}\phi^{(2)} - \left(\frac{3x_3^2}{h^3}-\frac{1}{h}\right)\phi^{(3)} \end{aligned} \tag{4.1}$$

As noted earlier, by virtue of the fact that in this low-order treatment we ignore the second-order equations completely, we do not need the plate strains $S_{ij}^{(2)}$ in $(3.10)_1$, which means that for our purposes here, eqn $(3.10)_1$ takes the form

$$S_{ij} = \sum_{n=0}^{1} x_3^n S_{ij}^{(n)} \tag{4.2}$$

which is directly analogous to (4.1).

Before proceeding, we introduce the usual compressed matrix notation for stress and strain in the constitutive relations, which will be convenient in what follows. In this convention the tensor indices ij or kl are replaced by p or q which take the values 1, 2, 3, 4, 5, 6 as ij or kl take the values 11, 22, 33, 23, or 32, 31 or 13, 12 or 21. Accordingly, we write the constitutive equations (2.2) in the compressed notation thus,

$$\begin{aligned} \tau_p &= c_{pq}S_q - e_{kp}E_k - \tfrac{1}{2}\hat{b}_{klp}E_kE_l - \tfrac{1}{6}d_{klmp}E_kE_lE_m \\ D_i &= e_{iq}S_q + \varepsilon_{ik}E_k + \tfrac{1}{2}\chi_{kji}E_kE_j + \tfrac{1}{6}\tilde{\chi}_{kjli}E_kE_jE_l \end{aligned} \tag{4.3}$$

where the sum from 1–6 on the repeated matrix indices is understood. We also note that in the matrix notation eqn (4.2) takes the form

$$S_q = \sum_{n=0}^{1} x_3^n S_q^{(n)} \tag{4.4}$$

Similarly, from eqn $(3.5)_1$ we write the relevant forms

$$\tau_p^{(0)} = \int_{-h}^{h} \tau_p \,\mathrm{d}x_3, \quad \tau_p^{(1)} = \int_{-h}^{h} x_3\tau_p \,\mathrm{d}x_3, \tag{4.5}$$

Now, substituting from (4.3) into (4.5) and $(3.5)_2$, introducing (4.4) and (4.1) and integrating through the thickness, we obtain the electroelastic plate constitutive equations in the form

$$\begin{aligned} \tau_p^{(0)} &= 2hc_{pq}S_q^{(0)} + 2he_{ap}\phi_{,a}^{(0)} - \tfrac{4}{3}he_{ap}\phi_{,a}^{(2)} + e_{3p}\phi^{(1)} \\ &\quad - \tfrac{1}{2}\hat{b}_{ijp}N_{ij}^{(0)} - \tfrac{1}{6}d_{ijkp}N_{ijk}^{(0)} \\ \tau_p^{(1)} &= \frac{2}{3}h^3c_{pq}S_q^{(1)} + \frac{h^2}{3}e_{ap}\phi_{,a}^{(1)} - \frac{4}{15}h^2e_{ap}\phi_{,a}^{(3)} + \frac{4}{3}he_{3p}\phi^{(2)} \\ &\quad - \frac{1}{2}\hat{b}_{ijp}N_{ij}^{(1)} - \frac{1}{6}d_{ijkp}N_{ijk}^{(1)} \end{aligned} \tag{4.6}$$

$$\begin{aligned} D_k^{(0)} &= 2he_{kp}S_p^{(0)} - 2h\varepsilon_{ka}\phi_{,a}^{(0)} + \tfrac{4}{3}h\varepsilon_{ka}\phi_{,a}^{(2)} - \varepsilon_{k3}\phi^{(1)} \\ &\quad + \tfrac{1}{2}\chi_{ijk}N_{ij}^{(0)} + \tfrac{1}{6}\tilde{\chi}_{ijmk}N_{ijm}^{(0)} \\ D_k^{(1)} &= \tfrac{2}{3}h^3e_{kp}S_p^{(1)} - \tfrac{1}{3}h^2\varepsilon_{ka}\phi_{,a}^{(1)} + \tfrac{4}{15}h^2\varepsilon_{ka}\phi_{,a}^{(3)} - \tfrac{4}{3}h\varepsilon_{k3}\phi^{(2)} \\ &\quad + \tfrac{1}{2}\chi_{ijk}N_{ij}^{(1)} + \tfrac{1}{6}\tilde{\chi}_{ijmk}N_{ijm}^{(1)} \\ D_k^{(2)} &= \tfrac{2}{3}h^3e_{kp}S_p^{(0)} - \tfrac{2}{3}h^3\varepsilon_{ka}\phi_{,a}^{(0)} + \tfrac{4}{15}h^3\varepsilon_{ka}\phi_{,a}^{(2)} - \tfrac{1}{3}h^2\varepsilon_{k3}\phi^{(1)} - \tfrac{8}{15}h^2\varepsilon_{k3}\phi^{(3)} \\ &\quad + \tfrac{1}{2}\chi_{ijk}N_{ij}^{(2)} + \tfrac{1}{6}\tilde{\chi}_{ijmk}N_{ijm}^{(2)} \\ D_k^{(3)} &= \tfrac{2}{5}h^5e_{kp}S_p^{(1)} - \tfrac{1}{5}h^4\varepsilon_{ka}\phi_{,a}^{(1)} + \tfrac{4}{35}h^4\varepsilon_{ka}\phi_{,a}^{(3)} - \tfrac{4}{5}h^3\varepsilon_{k3}\phi^{(2)} \\ &\quad + \tfrac{1}{2}\chi_{ijk}N_{ij}^{(3)} + \tfrac{1}{6}\tilde{\chi}_{ijmk}N_{ijm}^{(3)} \end{aligned} \tag{4.7}$$

where the nonlinear terms $N_{ij}^{(n)}$ and $N_{ijk}^{(n)}$ $(n = 0, 1, 2, 3)$ are given in Appendix I. These constitutive relations are not yet in a useful form for our purposes because we have not yet imposed the extensional plate relaxation conditions $(3.13)_1$ and flexural plate relaxation conditions $(3.15)_1$, which in the compressed matrix notation take the respective forms

$$\tau_3^{(0)} = 0, \quad \tau_4^{(0)} = 0, \quad \tau_5^{(0)} = 0 \tag{4.8}$$

$$\tau_3^{(1)} = 0, \quad \tau_4^{(1)} = 0, \quad \tau_5^{(1)} = 0 \tag{4.9}$$

In view of the conditions in (4.8) and (4.9) we now introduce the matrix index convention of Tiersten (1993b) which will be of considerable use to us in the sequel. The convention is that we let the subscripts u, v, w take the values 3, 4, 5 while the subscripts r, s, t take the remaining values 1, 2, 6. This permits us to write (4.8) and (4.9) in the form

$$\tau_w^{(0)} = 0, \quad \tau_w^{(1)} = 0 \tag{4.10}$$

We may now write eqn (4.6) in the form

$$\tau_r^{(0)} = 2hc_{rs}S_s^{(0)} + 2hc_{rv}S_v^{(0)} + 2he_{ar}\phi_{,a}^{(0)} - \tfrac{4}{3}he_{ar}\phi_{,a}^{(2)} + e_{3r}\phi^{(1)} - \tfrac{1}{2}\hat{b}_{ijr}N_{ij}^{(0)} - \tfrac{1}{6}d_{ijkr}N_{ijk}^{(0)} \tag{4.11}$$

$$\tau_w^{(0)} = 2hc_{ws}S_s^{(0)} + 2hc_{wv}S_v^{(0)} + 2he_{aw}\phi_{,a}^{(0)} - \tfrac{4}{3}he_{aw}\phi_{,a}^{(2)} + e_{3w}\phi^{(1)} - \tfrac{1}{2}\hat{b}_{ijw}N_{ij}^{(0)} - \tfrac{1}{6}d_{ijkw}N_{ijk}^{(0)} = 0 \tag{4.12}$$

$$\tau_r^{(1)} = \frac{2}{3}h^3c_{rs}S_s^{(1)} + \frac{2}{3}h^3c_{rv}S_v^{(1)} + \frac{h^2}{3}e_{ar}\phi_{,a}^{(1)} - \frac{4}{15}h^2e_{ar}\phi_{,a}^{(3)} + \frac{4}{3}he_{3r}\phi^{(2)} - \frac{1}{2}\hat{b}_{ijr}N_{ij}^{(1)} - \frac{1}{6}d_{ijkr}N_{ijk}^{(1)} \tag{4.13}$$

$$\tau_w^{(1)} = \frac{2}{3}h^3c_{ws}S_s^{(1)} + \frac{2}{3}h^3c_{wv}S_v^{(1)} + \frac{h^2}{3}e_{aw}\phi_{,a}^{(1)} - \frac{4}{15}h^2e_{aw}\phi_{,a}^{(3)} + \frac{4}{3}he_{3w}\phi^{(2)} - \frac{1}{2}\hat{b}_{ijw}N_{ij}^{(1)} - \frac{1}{6}d_{ijkw}N_{ijk}^{(1)} = 0 \tag{4.14}$$

Equations (4.12) and (4.14) may be readily solved for $S_v^{(0)}$ and $S_v^{(1)}$, respectively, with the result

$$S_v^{(0)} = -c_{vw}^{-1}c_{ws}S_s^{(0)} - e_{aw}c_{wv}^{-1}\phi_{,a}^{(0)} + \frac{2}{3}e_{aw}c_{wv}^{-1}\phi_{,a}^{(2)} - \frac{1}{2h}e_{3w}c_{wv}^{-1}\phi^{(1)} + \frac{1}{4h}\hat{b}_{ijw}c_{wv}^{-1}N_{ij}^{(0)} + \frac{1}{12h}d_{ijkw}c_{wv}^{-1}N_{ijk}^{(0)} \tag{4.15}$$

$$S_v^{(1)} = -c_{vw}^{-1}c_{ws}S_s^{(1)} - \frac{1}{2h}c_{vw}^{-1}e_{aw}\phi_{,a}^{(1)} + \frac{2}{5h}c_{vw}^{-1}e_{aw}\phi_{,a}^{(3)} - \frac{2}{h^2}c_{vw}^{-1}e_{3w}\phi^{(2)} + \frac{3}{4h^3}\hat{b}_{ijw}c_{wv}^{-1}N_{ij}^{(1)} + \frac{3}{12h^3}d_{ijkw}c_{wv}^{-1}N_{ijk}^{(1)} \tag{4.16}$$

where the matrix sums are over the indices 3, 4, 5 as a result of the convention. Substituting from (4.15) and (4.16) into (4.11) and (4.13), respectively, we obtain

$$\tau_r^{(0)} = 2h\gamma_{rs}S_s^{(0)} + 2h\psi_{cr}\phi_{,c}^{(0)} - \frac{4h}{3}\psi_{cr}\phi_{,c}^{(2)} + \psi_{3r}\phi^{(1)} - \frac{1}{2}\hat{b}_{ijr}^{p}N_{ij}^{(0)} - \frac{1}{6}d_{ijkr}^{p}N_{ijk}^{(0)} \tag{4.17}$$

$$\tau_r^{(1)} = \frac{2h^3}{3}\gamma_{rs}S_s^{(1)} + \frac{h^2}{3}\psi_{cr}\phi_{,c}^{(1)} - \frac{4h^2}{15}\psi_{cr}\phi_{,c}^{(3)} + \frac{4h}{3}\psi_{3r}\phi^{(2)} - \frac{1}{2}\hat{b}_{ijr}^{p}N_{ij}^{(1)} - \frac{1}{6}d_{ijkr}^{p}N_{ijk}^{(1)} \tag{4.18}$$

where

$$\gamma_{rs} = c_{rs} - c_{rv}c_{vw}^{-1}c_{ws}, \quad \psi_{ks} = e_{ks} - e_{kw}c_{wv}^{-1}c_{vs}$$
$$\hat{b}_{ijs}^{p} = \hat{b}_{ijs} - \hat{b}_{ijw}c_{wv}^{-1}c_{vs}, \quad d_{ijks}^{p} = d_{ijks} - d_{ijkw}c_{wv}^{-1}c_{vs} \tag{4.19}$$

and we note that the γ_{rs} are the Voigt's anisotropic plate elastic constants, the ψ_{ks} are plate piezoelectric constants and the $\hat{b}_{ijs}^{p}$ are the effective plate electrostrictive constants. The constitutive relations in (4.17) and (4.18) are in the form required for use in the equations for anisotropic extension (3.14) and elementary flexure (3.17). At this point we note that the substitution of (3.16) into $(3.10)_2$ yields

$$S_{ab}^{(1)} = -u_{3,ab}^{(0)} \tag{4.20}$$

which is an important relation for the flexural equation. We now write the $S_q^{(n)}$ in each of the constitutive equations (4.7) as two separate terms, one containing $S_s^{(n)}$ and the other containing $S_v^{(n)}$ in accordance with our convention, and substitute from (4.15) and (4.16) into the appropriate equations, with the result

$$D_k^{(0)} = 2h\psi_{ks}S_s^{(0)} - 2h\zeta_{kc}\phi_{,c}^{(0)} + \frac{4h}{3}\zeta_{kc}\phi_{,c}^{(2)} - \zeta_{k3}\phi^{(1)} + \frac{1}{2}\chi_{ijk}^{p}N_{ij}^{(0)} + \frac{1}{6}\tilde{\chi}_{ijmk}^{p}N_{ijm}^{(0)}$$

$$D_k^{(1)} = \frac{2h^3}{3}\psi_{ks}S_s^{(1)} - \frac{h^2}{3}\zeta_{kc}\phi_{,c}^{(1)} + \frac{4h^2}{15}\zeta_{kc}\phi_{,c}^{(3)} - \frac{4h}{3}\zeta_{k3}\phi^{(2)} + \frac{1}{2}\chi_{ijk}^{p}N_{ij}^{(1)} + \frac{1}{6}\tilde{\chi}_{ijmk}^{p}N_{ijm}^{(1)}$$

$$D_k^{(2)} = \frac{2h^3}{3}\psi_{ks}S_s^{(0)} - \frac{2}{3}h^3\zeta_{kc}\phi_{,c}^{(0)} + \frac{4h^3}{45}\hat{\zeta}_{kc}\phi_{,c}^{(2)} - \frac{h^2}{3}\zeta_{k3}\phi^{(1)} - \frac{8h^2}{15}\varepsilon_{k3}\phi^{(3)} + \frac{h^2}{6}\Delta_{ijk}N_{ij}^{(0)} + \frac{1}{2}\chi_{ijk}N_{ij}^{(2)} + \frac{h^2}{18}\Gamma_{ijmk}N_{ijm}^{(0)} + \frac{1}{6}\tilde{\chi}_{ijmk}N_{ijm}^{(2)}$$

$$D_k^{(3)} = \frac{2h^5}{5}\psi_{ks}S_s^{(1)} - \frac{h^4}{5}\zeta_{kc}\phi_{,c}^{(1)} + \frac{4h^4}{175}\bar{\zeta}_{kc}\phi_{,c}^{(3)} - \frac{4h^3}{5}\zeta_{k3}\phi^{(2)} + \frac{3h^2}{10}\Delta_{ijk}N_{ij}^{(1)} + \frac{1}{2}\chi_{ijk}N_{ij}^{(3)} + \frac{h^2}{10}\Gamma_{ijmk}N_{ijm}^{(1)} + \frac{1}{6}\tilde{\chi}_{ijmk}N_{ijm}^{(3)} \tag{4.21}$$

where

$$\zeta_{kj} = \varepsilon_{kj} + e_{kv} c_{vw}^{-1} e_{jw}, \quad \hat{\zeta}_{kc} = 3\varepsilon_{kc} + 5 e_{kv} c_{vw}^{-1} e_{cw}$$
$$\bar{\zeta}_{kc} = 5\varepsilon_{kc} + 7 e_{kv} c_{vw}^{-1} e_{cw}, \quad \chi_{ijk}^{p} = \chi_{ijk} + \Delta_{ijk}$$
$$\Delta_{ijk} = \hat{b}_{ijw} c_{wv}^{-1} e_{kv}, \quad \tilde{\chi}_{ijmk}^{p} = \tilde{\chi}_{ijmk} + \Gamma_{ijmk}$$
$$\Gamma_{ijmk} = d_{ijmw} c_{wv}^{-1} e_{kv} \tag{4.22}$$

The constitutive equations in (4.21) are in the form required for use in the plate electrostatic equation in (3.9). However, in order to use the constitutive equations in (4.17), (4.18) and (4.21) in the field equations (3.14), (3.17) and (3.9), they must be converted from matrix form back into tensor form. This is accomplished simply by replacing r by ab and s by cd whenever they occur in (4.17), (4.18) and (4.21). For example, (4.17) may be written

$$\tau_{ab}^{(0)} = 2h\gamma_{abcd} S_{cd}^{(0)} + 2h\psi_{cab}\phi_{,c}^{(0)} - \frac{4h}{3}\psi_{cab}\phi_{,c}^{(2)} + \psi_{3ab}\phi^{(1)} - \frac{1}{2}\hat{b}_{ijab}^{p} N_{ij}^{(0)} - \frac{1}{6} d_{ijkab}^{p} N_{ijk}^{(0)} \tag{4.23}$$

The substitution of the tensor form of the constitutive equations (4.17), (4.18) and (4.21) in (3.9), (3.14) and (3.17) along with $(3.10)_2$ for $S_{ab}^{(0)}$ and (4.20) yields seven equations in the seven dependent variables $u_j^{(0)}$, $\phi^{(0)}$, $\phi^{(1)}$, $\phi^{(2)}$, $\phi^{(3)}$, which are nonlinear in the plate electric potentials $\phi^{(n)}$. By appropriate substitutions the boundary conditions may be written in terms of the same seven variables in each region.

5. Summary of equations

Equations of balance (3.9), (3.14) and (3.17)

$$\tau_{ab,a}^{(0)} + F_b^{(0)} = 2\rho h \ddot{u}_b^{(0)}$$
$$\tau_{ab,ab}^{(1)} + F_{b,b}^{(1)} + F_3^{(0)} = 2\rho h \ddot{u}_3^{(0)}$$
$$D_{a,a}^{(0)} + 2h d_3^{(0)} = 0$$
$$\frac{1}{2h} D_{a,a}^{(1)} - \frac{1}{2h} D_3^{(0)} + d_3^{(1)} = 0$$
$$\frac{1}{h^2} D_{a,a}^{(2)} - D_{a,a}^{(0)} - \frac{2}{h^2} D_3^{(1)} = 0$$
$$\frac{1}{h^3} D_{a,a}^{(3)} - \frac{1}{h} D_{a,a}^{(1)} - \frac{3}{h^3} D_3^{(2)} + \frac{1}{h} D_3^{(0)} = 0 \tag{5.1}$$

Strain–displacement relations $(3.10)_2$ and (4.20)

$$S_{ab}^{(0)} = \tfrac{1}{2}(u_{a,b}^{(0)} + u_{b,a}^{(0)})$$
$$S_{ab}^{(1)} = -u_{3,ab}^{(0)} \tag{5.2}$$

Constitutive equations (4.17), (4.18) and (4.21)

$$
\begin{aligned}
\tau_{ab}^{(0)} &= 2h\gamma_{abcd}S_{cd}^{(0)} + 2h\psi_{cab}\phi_{,c}^{(0)} - \frac{4h}{3}\psi_{cab}\phi_{,c}^{(2)} + \psi_{3ab}\phi^{(1)} \\
&\quad - \frac{1}{2}\hat{b}_{ijab}^{p}N_{ij}^{(0)} - \frac{1}{6}d_{ijkab}^{p}N_{ijk}^{(0)} \\
\tau_{ab}^{(1)} &= \frac{2h^3}{3}\gamma_{abcd}S_{cd}^{(1)} + \frac{h^2}{3}\psi_{cab}\phi_{,c}^{(1)} - \frac{4h^2}{15}\psi_{cab}\phi_{,c}^{(3)} + \frac{4h}{3}\psi_{3ab}\phi^{(2)} \\
&\quad - \frac{1}{2}\hat{b}_{ijab}^{p}N_{ij}^{(1)} - \frac{1}{6}d_{ijkab}^{p}N_{ijk}^{(1)} \\
D_{k}^{(0)} &= 2h\psi_{kcd}S_{cd}^{(0)} - 2h\zeta_{kc}\psi_{,c}^{(0)} + \frac{4h}{3}\zeta_{kc}\phi_{,c}^{(2)} - \zeta_{k3}\phi^{(1)} \\
&\quad + \frac{1}{2}\chi_{ijk}^{p}N_{ij}^{(0)} + \frac{1}{6}\tilde{\chi}_{ijmk}^{p}N_{ijm}^{(0)} \\
D_{k}^{(1)} &= \frac{2h^3}{3}\psi_{kcd}S_{cd}^{(1)} - \frac{h^2}{3}\zeta_{kc}\phi_{,c}^{(1)} + \frac{4h^2}{15}\zeta_{kc}\phi_{,c}^{(3)} - \frac{4h}{3}\zeta_{k3}\phi^{(2)} \\
&\quad + \frac{1}{2}\chi_{ijk}^{p}N_{ij}^{(1)} + \frac{1}{6}\tilde{\chi}_{ijmk}^{p}N_{ijm}^{(1)} \\
D_{k}^{(2)} &= \frac{2h^3}{3}\psi_{kcd}S_{cd}^{(0)} - \frac{2}{3}h^3\zeta_{kc}\phi_{,c}^{(0)} + \frac{4h^3}{45}\hat{\zeta}_{kc}\phi_{,c}^{(2)} - \frac{h^2}{3}\zeta_{k3}\phi^{(1)} - \frac{8h^2}{15}\varepsilon_{k3}\phi^{(3)} \\
&\quad + \frac{h^2}{6}\Delta_{ijk}N_{ij}^{(0)} + \frac{1}{2}\chi_{ijk}N_{ij}^{(2)} + \frac{h^2}{18}\Gamma_{ijmk}N_{ijm}^{(0)} + \frac{1}{6}\tilde{\chi}_{ijmk}N_{ijm}^{(2)} \\
D_{k}^{(3)} &= \frac{2h^5}{5}\psi_{kcd}S_{cd}^{(1)} - \frac{h^4}{5}\zeta_{kc}\phi_{,c}^{(1)} + \frac{4h^4}{175}\bar{\zeta}_{kc}\phi_{,c}^{(3)} - \frac{4h^3}{5}\zeta_{k3}\phi^{(2)} \\
&\quad + \frac{3h^2}{10}\Delta_{ijk}N_{ij}^{(1)} + \frac{1}{2}\chi_{ijk}N_{ij}^{(3)} + \frac{h^2}{10}\Gamma_{ijmk}N_{ijm}^{(1)} + \frac{1}{6}\tilde{\chi}_{ijmk}N_{ijm}^{(3)}
\end{aligned}
\tag{5.3}
$$

Continuity conditions (3.18), (3.19), (3.21) and (3.22)

$$
\begin{aligned}
& n_a[\![\tau_{ab}^{(0)}]\!] = 0, \quad [\![u_b^{(0)}]\!] = 0 \\
& n_a[\![\tau_{ab}^{(1)}]\!]n_b = 0, \quad \left[\!\!\left[\tau_{n3}^{(0)} + \frac{\partial \tau_{ns}^{(1)}}{\partial s} \right]\!\!\right] = 0, \quad \left[\!\!\left[\frac{\partial u_3^{(0)}}{\partial n} \right]\!\!\right] = 0, \quad [\![u_3^{(0)}]\!] = 0 \\
& [\![\phi^{(n)}]\!] = 0, \quad n = 0, 1, 2, 3 \\
& n_a[\![D_a^{(0)}]\!] = 0, \quad n_a[\![D_a^{(1)}]\!] = 0 \\
& n_a[\![D_a^{(2)} - h^2 D_a^{(0)}]\!] = 0, \quad n_a[\![D_a^{(3)} - h^2 D_a^{(1)}]\!] = 0
\end{aligned}
\tag{5.4}
$$

Definitions of material constants (4.19) and (4.22)

$$\gamma_{rs} = c_{rs} - c_{rv}c_{vw}^{-1}c_{ws}, \quad \psi_{ks} = e_{ks} - e_{kw}c_{wv}^{-1}c_{vs}$$
$$\hat{b}_{ijs}^{p} = \hat{b}_{ijs} - \hat{b}_{ijw}c_{wv}^{-1}c_{vs}, \quad d_{ijks}^{p} = d_{ijks} - d_{ijkw}c_{wv}^{-1}c_{vs}$$
$$\zeta_{kj} = \varepsilon_{kj} + e_{kv}c_{vw}^{-1}e_{jw}, \quad \hat{\zeta}_{kc} = 3\varepsilon_{kc} + 5e_{kv}c_{vw}^{-1}e_{cw}$$
$$\bar{\zeta}_{kc} = 5\varepsilon_{kc} + 7e_{kv}c_{vw}^{-1}e_{cw}, \quad \chi_{ijk}^{p} = \chi_{ijk} + \Delta_{ijk}$$
$$\Delta_{ijk} = \hat{b}_{ijw}c_{wv}^{-1}e_{kv}, \quad \tilde{\chi}_{ijmk}^{p} = \tilde{\chi}_{ijmk} + \Gamma_{ijmk}$$
$$\Gamma_{ijmk} = d_{ijmw}c_{wv}^{-1}e_{kv} \tag{5.5}$$

Ranges of indices

$$i,j,k,l,m,n: \quad 1,2,3; a,b,c,d: \quad 1,2; p,q: \quad 1,2,3,4,5,6; r,s,t: \quad 1,2,6; u,v,w: \quad 3,4,5$$

6. Conclusions

A set of two-dimensional plate equations are derived for the extension and flexure of relatively thin electroelastic plates under strong electric fields. The equations are cubic in the electric potentials and can be applied to electroded or unelectroded plates. The equations derived above include the results of Tiersten (1993b) as special cases.

Appendix I: Nonlinear electrical terms

$$N_{ab}^{(0)} = h[2\phi_{,a}^{(0)}\phi_{,b}^{(0)} + \tfrac{16}{105}\phi_{,a}^{(3)}\phi_{,b}^{(3)} - \tfrac{4}{3}(\phi_{,a}^{(2)}\phi_{,b}^{(0)} + \phi_{,b}^{(2)}\phi_{,a}^{(0)})$$
$$- \tfrac{2}{15}(\phi_{,a}^{(1)}\phi_{,b}^{(3)} + \phi_{,a}^{(3)}\phi_{,b}^{(1)}) + \tfrac{16}{15}\phi_{,a}^{(2)}\phi_{,b}^{(2)} + \tfrac{1}{6}\phi_{,a}^{(1)}\phi_{,b}^{(1)}]$$
$$N_{3b}^{(0)} = \tfrac{8}{15}(\phi^{(3)}\phi_{,b}^{(2)} - \phi^{(2)}\phi_{,b}^{(3)}) + \tfrac{2}{3}(\phi^{(2)}\phi_{,b}^{(1)} - \phi^{(1)}\phi_{,b}^{(2)}) + \phi^{(1)}\phi_{,b}^{(0)}$$
$$N_{33}^{(0)} = \frac{1}{30h}[48\phi^{(3)}\phi^{(3)} + 80\phi^{(2)}\phi^{(2)} + 15\phi^{(1)}\phi^{(1)}]$$
$$N_{ab}^{(1)} = h^2[\tfrac{16}{105}(\phi_{,a}^{(2)}\phi_{,b}^{(3)} + \phi_{,a}^{(3)}\phi_{,b}^{(2)}) - \tfrac{4}{15}(\phi_{,a}^{(3)}\phi_{,b}^{(0)} + \phi_{,b}^{(3)}\phi_{,a}^{(0)})$$
$$- \tfrac{2}{15}(\phi_{,a}^{(2)}\phi_{,b}^{(1)} + \phi_{,b}^{(2)}\phi_{,a}^{(1)}) + \tfrac{1}{3}(\phi_{,a}^{(0)}\phi_{,b}^{(1)} + \phi_{,b}^{(0)}\phi_{,a}^{(1)})]$$
$$N_{3b}^{(1)} = h[\tfrac{1}{6}\phi^{(1)}\phi_{,b}^{(1)} + \tfrac{4}{15}\phi^{(3)}\phi_{,b}^{(1)} - \tfrac{8}{15}\phi^{(2)}\phi_{,b}^{(2)} - \tfrac{2}{15}\phi^{(1)}\phi_{,b}^{(3)} - \tfrac{8}{105}\phi^{(3)}\phi_{,b}^{(3)} + \tfrac{4}{3}\phi^{(2)}\phi_{,b}^{(0)}]$$
$$N_{33}^{(1)} = \tfrac{4}{3}\phi^{(2)}\phi^{(1)} + \tfrac{32}{15}\phi^{(2)}\phi^{(3)}$$
$$N_{ab}^{(2)} = h^3[\tfrac{16}{315}\phi_{,a}^{(3)}\phi_{,b}^{(3)} - \tfrac{4}{15}(\phi_{,a}^{(2)}\phi_{,b}^{(0)} + \phi_{,b}^{(2)}\phi_{,a}^{(0)}) - \tfrac{2}{35}(\phi_{,a}^{(3)}\phi_{,b}^{(1)} + \phi_{,b}^{(3)}\phi_{,a}^{(1)})$$
$$+ \tfrac{16}{105}\phi_{,a}^{(2)}\phi_{,b}^{(2)} + \tfrac{1}{10}\phi_{,a}^{(1)}\phi_{,b}^{(1)} + \tfrac{2}{3}\phi_{,a}^{(0)}\phi_{,b}^{(0)}]$$
$$N_{3b}^{(2)} = h^2[\tfrac{8}{15}\phi^{(3)}\phi_{,b}^{(0)} + \tfrac{2}{5}\phi^{(2)}\phi_{,b}^{(1)} - \tfrac{2}{15}\phi^{(1)}\phi_{,b}^{(2)} + \tfrac{1}{3}\phi^{(1)}\phi_{,b}^{(0)} - \tfrac{8}{35}\phi^{(2)}\phi_{,b}^{(3)} - \tfrac{8}{105}\phi^{(3)}\phi_{,b}^{(2)}]$$
$$N_{33}^{(2)} = h[\tfrac{8}{15}\phi^{(3)}\phi^{(1)} + \tfrac{88}{105}\phi^{(3)}\phi^{(3)} + \tfrac{1}{6}\phi^{(1)}\phi^{(1)} + \tfrac{8}{5}\phi^{(2)}\phi^{(2)}]$$

$$N_{ab}^{(3)} = h^4[\tfrac{16}{315}(\phi_{,a}^{(3)}\phi_{,b}^{(2)}+\phi_{,b}^{(3)}\phi_{,a}^{(2)})-\tfrac{4}{45}(\phi_{,a}^{(0)}\phi_{,b}^{(3)}+\phi_{,b}^{(0)}\phi_{,a}^{(3)})$$
$$-\tfrac{2}{35}(\phi_{,a}^{(2)}\phi_{,b}^{(1)}+\phi_{,b}^{(2)}\phi_{,a}^{(1)})+\tfrac{1}{5}(\phi_{,a}^{(0)}\phi_{,b}^{(1)}+\phi_{,b}^{(0)}\phi_{,a}^{(1)})]$$

$$N_{3b}^{(3)} = h^3[\tfrac{1}{10}\phi^{(1)}\phi_{,b}^{(1)}+\tfrac{8}{35}(\phi^{(3)}\phi_{,b}^{(1)}-\phi^{(2)}\phi_{,b}^{(2)})-\tfrac{2}{35}\phi^{(1)}\phi_{,b}^{(3)}-\tfrac{8}{105}\phi^{(3)}\phi_{,b}^{(3)}+\tfrac{4}{5}\phi^{(2)}\phi_{,b}^{(0)}]$$
$$N_{33}^{(3)} = h^2[\tfrac{4}{5}\phi^{(2)}\phi^{(1)}+\tfrac{64}{35}\phi^{(2)}\phi^{(3)}]$$
$$N_{abc}^{(0)} = h[\tfrac{2}{15}(\phi_{,a}^{(3)}\phi_{,b}^{(0)}\phi_{,c}^{(1)}+\phi_{,b}^{(3)}\phi_{,c}^{(0)}\phi_{,a}^{(1)}+\phi_{,a}^{(3)}\phi_{,c}^{(0)}\phi_{,b}^{(1)}$$
$$+\phi_{,c}^{(3)}\phi_{,a}^{(0)}\phi_{,b}^{(1)}+\phi_{,c}^{(3)}\phi_{,b}^{(0)}\phi_{,a}^{(1)}+\phi_{,b}^{(3)}\phi_{,a}^{(0)}\phi_{,c}^{(1)})$$
$$-\tfrac{16}{15}(\phi_{,a}^{(2)}\phi_{,b}^{(2)}\phi_{,c}^{(0)}+\phi_{,c}^{(2)}\phi_{,b}^{(2)}\phi_{,a}^{(0)}+\phi_{,a}^{(2)}\phi_{,c}^{(2)}\phi_{,b}^{(0)})$$
$$+\tfrac{1}{15}(\phi_{,a}^{(2)}\phi_{,b}^{(1)}\phi_{,c}^{(1)}+\phi_{,c}^{(2)}\phi_{,b}^{(1)}\phi_{,a}^{(1)}+\phi_{,b}^{(2)}\phi_{,c}^{(1)}\phi_{,a}^{(1)})$$
$$-\tfrac{1}{6}(\phi_{,a}^{(1)}\phi_{,b}^{(0)}\phi_{,c}^{(1)}+\phi_{,a}^{(1)}\phi_{,c}^{(0)}\phi_{,b}^{(1)}+\phi_{,b}^{(1)}\phi_{,a}^{(0)}\phi_{,c}^{(1)})$$
$$+\tfrac{4}{3}(\phi_{,c}^{(2)}\phi_{,a}^{(0)}\phi_{,b}^{(0)}+\phi_{,a}^{(2)}\phi_{,c}^{(0)}\phi_{,b}^{(0)}+\phi_{,b}^{(2)}\phi_{,c}^{(0)}\phi_{,a}^{(0)})-2\phi_{,a}^{(0)}\phi_{,b}^{(0)}\phi_{,c}^{(0)}$$
$$-\tfrac{8}{105}(\phi_{,a}^{(3)}\phi_{,b}^{(2)}\phi_{,c}^{(1)}+\phi_{,c}^{(3)}\phi_{,a}^{(2)}\phi_{,b}^{(1)}+\phi_{,b}^{(3)}\phi_{,c}^{(2)}\phi_{,a}^{(1)}$$
$$+\phi_{,c}^{(3)}\phi_{,b}^{(2)}\phi_{,a}^{(1)}+\phi_{,b}^{(3)}\phi_{,a}^{(2)}\phi_{,c}^{(1)}+\phi_{,a}^{(3)}\phi_{,c}^{(2)}\phi_{,b}^{(1)})$$
$$-\tfrac{16}{105}(\phi_{,c}^{(3)}\phi_{,a}^{(3)}\phi_{,b}^{(0)}+\phi_{,c}^{(3)}\phi_{,b}^{(3)}\phi_{,a}^{(0)}+\phi_{,a}^{(3)}\phi_{,b}^{(3)}\phi_{,c}^{(0)})$$
$$+\tfrac{32}{315}(\phi_{,a}^{(3)}\phi_{,b}^{(3)}\phi_{,c}^{(2)}+\phi_{,c}^{(3)}\phi_{,b}^{(3)}\phi_{,a}^{(2)}+\phi_{,c}^{(3)}\phi_{,a}^{(3)}\phi_{,b}^{(2)})+\tfrac{32}{35}\phi_{,a}^{(2)}\phi_{,b}^{(2)}\phi_{,c}^{(2)}]$$

$$N_{ab3}^{(0)} = \tfrac{4}{105}(\phi^{(3)}\phi_{,b}^{(3)}\phi_{,a}^{(1)}+\phi^{(3)}\phi_{,a}^{(3)}\phi_{,b}^{(1)})+\tfrac{64}{105}\phi^{(3)}\phi_{,a}^{(2)}\phi_{,b}^{(2)}$$
$$-\tfrac{32}{105}(\phi^{(2)}\phi_{,a}^{(2)}\phi_{,b}^{(3)}+\phi^{(2)}\phi_{,a}^{(3)}\phi_{,b}^{(2)})-\tfrac{8}{105}\phi_{,b}^{(3)}\phi_{,a}^{(3)}\phi^{(1)}$$
$$+\tfrac{1}{15}(\phi_{,a}^{(3)}\phi_{,b}^{(1)}\phi^{(1)}+\phi_{,b}^{(3)}\phi_{,a}^{(1)}\phi^{(1)})+\tfrac{8}{15}(-\phi^{(3)}\phi_{,b}^{(2)}\phi_{,a}^{(0)}$$
$$-\phi^{(3)}\phi_{,a}^{(2)}\phi_{,b}^{(0)}+\phi^{(2)}\phi_{,b}^{(3)}\phi_{,a}^{(0)}-\phi^{(1)}\phi_{,b}^{(2)}\phi_{,a}^{(2)}+\phi^{(2)}\phi_{,a}^{(3)}\phi_{,b}^{(0)})-\tfrac{2}{15}\phi^{(3)}\phi_{,a}^{(1)}\phi_{,b}^{(1)}$$
$$+\tfrac{4}{15}(\phi^{(2)}\phi_{,b}^{(2)}\phi_{,a}^{(1)}+\phi^{(2)}\phi_{,a}^{(2)}\phi_{,b}^{(1)})-\phi_{,a}^{(0)}\phi_{,b}^{(0)}\phi^{(1)}-\tfrac{1}{12}\phi_{,a}^{(1)}\phi_{,b}^{(1)}\phi^{(1)}$$
$$+\tfrac{2}{3}(-\phi^{(2)}\phi_{,a}^{(0)}\phi_{,b}^{(1)}-\phi^{(2)}\phi_{,b}^{(0)}\phi_{,a}^{(1)}+\phi^{(1)}\phi_{,b}^{(0)}\phi_{,a}^{(2)}+\phi^{(1)}\phi_{,a}^{(0)}\phi_{,b}^{(2)})]$$
$$N_{a33}^{(0)} = -\frac{1}{210h}(140\phi^{(2)}\phi_{,a}^{(1)}\phi^{(1)}-160\phi_{,a}^{(2)}\phi^{(3)}\phi^{(3)}+224\phi^{(2)}\phi^{(3)}\phi_{,a}^{(1)}+112\phi_{,a}^{(2)}\phi^{(3)}\phi^{(1)}$$
$$-112\phi_{,a}^{(3)}\phi^{(2)}\phi^{(1)}+560\phi_{,a}^{(0)}\phi^{(2)}\phi^{(2)}-70\phi_{,a}^{(2)}\phi^{(1)}\phi^{(1)}+336\phi_{,a}^{(0)}\phi^{(3)}\phi^{(3)}$$
$$-224\phi_{,a}^{(2)}\phi^{(2)}\phi^{(2)}-64\phi_{,a}^{(3)}\phi^{(2)}\phi^{(3)}+105\phi_{,a}^{(0)}\phi^{(1)}\phi^{(1)})$$

$$N_{333}^{(0)} = -\frac{1}{140h^2}(896\phi^{(2)}\phi^{(2)}\phi^{(3)}+128\phi^{(3)}\phi^{(3)}\phi^{(3)}+560\phi^{(1)}\phi^{(2)}\phi^{(2)}$$

$$+336\phi^{(1)}\phi^{(3)}\phi^{(3)}+35\phi^{(1)}\phi^{(1)}\phi^{(1)})$$

$$\begin{aligned}
N^{(1)}_{abc} = h^2[&-\tfrac{16}{105}(\phi^{(3)}_{,c}\phi^{(2)}_{,b}\phi^{(0)}_{,a}+\phi^{(3)}_{,a}\phi^{(2)}_{,c}\phi^{(0)}_{,b}+\phi^{(3)}_{,c}\phi^{(2)}_{,a}\phi^{(0)}_{,b}\\
&+\phi^{(3)}_{,b}\phi^{(2)}_{,c}\phi^{(0)}_{,a}+\phi^{(3)}_{,b}\phi^{(2)}_{,a}\phi^{(0)}_{,c}+\phi^{(3)}_{,a}\phi^{(2)}_{,b}\phi^{(0)}_{,c})\\
&+\tfrac{32}{315}(\phi^{(3)}_{,a}\phi^{(2)}_{,c}\phi^{(2)}_{,b}+\phi^{(3)}_{,c}\phi^{(2)}_{,a}\phi^{(2)}_{,b}+\phi^{(3)}_{,b}\phi^{(2)}_{,a}\phi^{(2)}_{,c})+\tfrac{32}{1155}\phi^{(3)}_{,c}\phi^{(3)}_{,b}\phi^{(3)}_{,a}\\
&-\tfrac{8}{315}(\phi^{(3)}_{,c}\phi^{(3)}_{,b}\phi^{(1)}_{,a}+\phi^{(3)}_{,c}\phi^{(3)}_{,a}\phi^{(1)}_{,b}+\phi^{(3)}_{,b}\phi^{(3)}_{,a}\phi^{(1)}_{,c})\\
&+\tfrac{2}{15}(\phi^{(2)}_{,c}\phi^{(0)}_{,a}\phi^{(1)}_{,b}+\phi^{(2)}_{,c}\phi^{(0)}_{,b}\phi^{(1)}_{,a}+\phi^{(2)}_{,b}\phi^{(0)}_{,a}\phi^{(1)}_{,c}\\
&+\phi^{(2)}_{,a}\phi^{(0)}_{,b}\phi^{(1)}_{,c}+\phi^{(2)}_{,b}\phi^{(0)}_{,c}\phi^{(1)}_{,a}+\phi^{(2)}_{,a}\phi^{(0)}_{,c}\phi^{(1)}_{,b})\\
&+\tfrac{4}{15}(\phi^{(3)}_{,a}\phi^{(0)}_{,b}\phi^{(0)}_{,c}+\phi^{(3)}_{,c}\phi^{(0)}_{,b}\phi^{(0)}_{,a}+\phi^{(3)}_{,b}\phi^{(0)}_{,c}\phi^{(0)}_{,a})\\
&-\tfrac{1}{3}(\phi^{(0)}_{,a}\phi^{(0)}_{,b}\phi^{(1)}_{,c}+\phi^{(0)}_{,a}\phi^{(0)}_{,c}\phi^{(1)}_{,b}+\phi^{(0)}_{,b}\phi^{(0)}_{,c}\phi^{(1)}_{,a})\\
&+\tfrac{1}{35}(\phi^{(3)}_{,c}\phi^{(1)}_{,a}\phi^{(1)}_{,b}+\phi^{(3)}_{,a}\phi^{(1)}_{,c}\phi^{(1)}_{,b}+\phi^{(3)}_{,b}\phi^{(1)}_{,c}\phi^{(1)}_{,a})\\
&-\tfrac{8}{105}(\phi^{(2)}_{,c}\phi^{(2)}_{,b}\phi^{(1)}_{,a}+\phi^{(2)}_{,c}\phi^{(2)}_{,a}\phi^{(1)}_{,b}+\phi^{(2)}_{,b}\phi^{(2)}_{,a}\phi^{(1)}_{,c})-\tfrac{1}{20}\phi^{(1)}_{,a}\phi^{(1)}_{,b}\phi^{(1)}_{,c}]
\end{aligned}$$

$$\begin{aligned}
N^{(1)}_{ab3} = h[&-\tfrac{32}{315}\phi^{(3)}_{,a}\phi^{(3)}_{,b}\phi^{(2)}+\tfrac{4}{35}(\phi^{(2)}\phi^{(3)}_{,a}\phi^{(1)}_{,b}+\phi^{(2)}\phi^{(3)}_{,b}\phi^{(1)}_{,a})\\
&+\tfrac{2}{15}(\phi^{(3)}_{,b}\phi^{(0)}_{,a}\phi^{(1)}+\phi^{(3)}_{,a}\phi^{(0)}_{,b}\phi^{(1)})\\
&+\tfrac{8}{105}(\phi^{(3)}\phi^{(3)}_{,a}\phi^{(0)}_{,b}+\phi^{(3)}\phi^{(3)}_{,b}\phi^{(0)}_{,a}-\phi^{(1)}\phi^{(3)}_{,b}\phi^{(2)}_{,a}-\phi^{(1)}\phi^{(3)}_{,a}\phi^{(2)}_{,b})\\
&-\tfrac{32}{105}\phi^{(2)}\phi^{(2)}_{,a}\phi^{(2)}_{,b}+\tfrac{1}{15}(\phi^{(2)}_{,b}\phi^{(1)}_{,a}\phi^{(1)}+\phi^{(2)}_{,a}\phi^{(1)}_{,b}\phi^{(1)})\\
&-\tfrac{1}{5}\phi^{(2)}\phi^{(1)}_{,a}\phi^{(1)}_{,b}+\tfrac{8}{15}(\phi^{(2)}\phi^{(2)}_{,a}\phi^{(0)}_{,b}+\phi^{(2)}\phi^{(2)}_{,b}\phi^{(0)}_{,a})-\tfrac{4}{15}(\phi^{(3)}\phi^{(0)}_{,a}\phi^{(1)}_{,b}+\phi^{(3)}\phi^{(0)}_{,b}\phi^{(1)}_{,a})\\
&-\tfrac{1}{6}(\phi^{(0)}_{,a}\phi^{(1)}_{,b}\phi^{(1)}+\phi^{(0)}_{,b}\phi^{(1)}_{,a}\phi^{(1)})+\tfrac{4}{105}(\phi^{(3)}\phi^{(2)}_{,b}\phi^{(1)}_{,a}+\phi^{(3)}\phi^{(2)}_{,a}\phi^{(1)}_{,b})-\tfrac{4}{3}\phi^{(2)}\phi^{(0)}_{,a}\phi^{(0)}_{,b}]
\end{aligned}$$

$$\begin{aligned}
N^{(1)}_{a33} = &\tfrac{8}{105}\phi^{(3)}_{,a}\phi^{(3)}\phi^{(1)}-\tfrac{4}{3}\phi^{(1)}\phi^{(2)}\phi^{(0)}_{,a}+\tfrac{1}{15}\phi^{(3)}_{,a}\phi^{(1)}\phi^{(1)}\\
&-\tfrac{4}{15}\phi^{(1)}_{,a}\phi^{(3)}\phi^{(1)}-\tfrac{32}{15}\phi^{(2)}\phi^{(3)}\phi^{(0)}_{,a}\\
&+\tfrac{8}{15}\phi^{(2)}_{,a}\phi^{(2)}\phi^{(1)}-\tfrac{4}{5}\phi^{(1)}_{,a}\phi^{(2)}\phi^{(2)}-\tfrac{44}{105}\phi^{(3)}\phi^{(3)}\phi^{(1)}_{,a}+\tfrac{16}{105}\phi^{(3)}\phi^{(3)}\phi^{(3)}_{,a}\\
&+\tfrac{16}{35}\phi^{(3)}_{,a}\phi^{(2)}\phi^{(2)}-\tfrac{1}{12}\phi^{(1)}_{,a}\phi^{(1)}\phi^{(1)}+\tfrac{32}{105}\phi^{(2)}_{,a}\phi^{(2)}\phi^{(3)}
\end{aligned}$$

$$N^{(1)}_{333} = -\frac{1}{35h}\phi^{(2)}(112\phi^{(2)}\phi^{(2)}+35\phi^{(1)}\phi^{(1)}+176\phi^{(3)}\phi^{(3)}+112\phi^{(3)}\phi^{(1)})$$

$$\begin{aligned}
N^{(2)}_{abc} = h^3[&\tfrac{4}{15}(\phi^{(2)}_{,b}\phi^{(0)}_{,a}\phi^{(0)}_{,c}+\phi^{(2)}_{,a}\phi^{(0)}_{,b}\phi^{(0)}_{,c}+\phi^{(2)}_{,c}\phi^{(0)}_{,b}\phi^{(0)}_{,a})\\
&-\tfrac{16}{315}(\phi^{(3)}_{,b}\phi^{(3)}_{,a}\phi^{(0)}_{,c}+\phi^{(3)}_{,c}\phi^{(3)}_{,a}\phi^{(0)}_{,b}+\phi^{(3)}_{,c}\phi^{(3)}_{,b}\phi^{(0)}_{,a})\\
&+\tfrac{32}{1155}(\phi^{(3)}_{,c}\phi^{(2)}_{,a}\phi^{(3)}_{,b}+\phi^{(3)}_{,a}\phi^{(2)}_{,c}\phi^{(3)}_{,b}+\phi^{(3)}_{,a}\phi^{(2)}_{,b}\phi^{(3)}_{,c})
\end{aligned}$$

$$
\begin{aligned}
&-\tfrac{1}{10}(\phi^{(1)}_{,a}\phi^{(1)}_{,b}\phi^{(0)}_{,c}+\phi^{(1)}_{,a}\phi^{(1)}_{,c}\phi^{(0)}_{,b}+\phi^{(1)}_{,b}\phi^{(1)}_{,c}\phi^{(0)}_{,a})\\
&-\tfrac{8}{315}(\phi^{(2)}_{,c}\phi^{(3)}_{,a}\phi^{(1)}_{,b}+\phi^{(2)}_{,b}\phi^{(3)}_{,a}\phi^{(1)}_{,c}+\phi^{(2)}_{,a}\phi^{(3)}_{,b}\phi^{(1)}_{,c}\\
&+\phi^{(2)}_{,c}\phi^{(3)}_{,b}\phi^{(1)}_{,a}+\phi^{(2)}_{,a}\phi^{(3)}_{,c}\phi^{(1)}_{,b}+\phi^{(2)}_{,b}\phi^{(3)}_{,c}\phi^{(1)}_{,a})\\
&+\tfrac{2}{35}(\phi^{(3)}_{,b}\phi^{(0)}_{,a}\phi^{(1)}_{,c}+\phi^{(3)}_{,a}\phi^{(0)}_{,c}\phi^{(1)}_{,b}+\phi^{(3)}_{,b}\phi^{(0)}_{,c}\phi^{(1)}_{,a}\\
&+\phi^{(3)}_{,a}\phi^{(0)}_{,b}\phi^{(1)}_{,c}+\phi^{(3)}_{,c}\phi^{(0)}_{,a}\phi^{(1)}_{,b}+\phi^{(3)}_{,c}\phi^{(0)}_{,b}\phi^{(1)}_{,a})\\
&-\tfrac{16}{105}(\phi^{(2)}_{,a}\phi^{(2)}_{,b}\phi^{(0)}_{,c}+\phi^{(2)}_{,a}\phi^{(2)}_{,c}\phi^{(0)}_{,b}+\phi^{(2)}_{,b}\phi^{(2)}_{,c}\phi^{(0)}_{,a})\\
&+\tfrac{1}{35}(\phi^{(2)}_{,a}\phi^{(1)}_{,b}\phi^{(1)}_{,c}+\phi^{(2)}_{,b}\phi^{(1)}_{,a}\phi^{(1)}_{,c}+\phi^{(2)}_{,c}\phi^{(1)}_{,a}\phi^{(1)}_{,b})\\
&-\tfrac{2}{3}\phi^{(0)}_{,a}\phi^{(0)}_{,b}\phi^{(0)}_{,c}+\tfrac{32}{315}\phi^{(2)}_{,a}\phi^{(2)}_{,b}\phi^{(2)}_{,c}]
\end{aligned}
$$

$$
\begin{aligned}
N^{(2)}_{ab3} = h^2[&\tfrac{2}{15}(\phi^{(2)}_{,a}\phi^{(0)}_{,b}\phi^{(1)}+\phi^{(2)}_{,b}\phi^{(0)}_{,a}\phi^{(1)})-\tfrac{8}{15}\phi^{(3)}\phi^{(0)}_{,a}\phi^{(0)}_{,b}\\
&-\tfrac{2}{5}(\phi^{(2)}\phi^{(0)}_{,a}\phi^{(1)}_{,b}+\phi^{(2)}\phi^{(0)}_{,b}\phi^{(1)}_{,a})\\
&-\tfrac{1}{3}\phi^{(0)}_{,a}\phi^{(0)}_{,b}\phi^{(1)}+\tfrac{8}{35}(\phi^{(2)}\phi^{(3)}_{,b}\phi^{(0)}_{,a}+\phi^{(2)}\phi^{(3)}_{,a}\phi^{(0)}_{,b})\\
&+\tfrac{4}{35}(\phi^{(2)}\phi^{(2)}_{,b}\phi^{(1)}_{,a}+\phi^{(2)}\phi^{(2)}_{,a}\phi^{(1)}_{,b}-\phi^{(3)}\phi^{(1)}_{,a}\phi^{(1)}_{,b})\\
&+\tfrac{1}{35}(\phi^{(3)}_{,b}\phi^{(1)}_{,a}\phi^{(1)}+\phi^{(3)}_{,a}\phi^{(1)}_{,b}\phi^{(1)})\\
&+\tfrac{8}{105}(-\phi^{(2)}_{,a}\phi^{(2)}_{,b}\phi^{(1)}+\phi^{(2)}_{,b}\phi^{(0)}_{,a}\phi^{(3)}+\phi^{(2)}_{,a}\phi^{(0)}_{,b}\phi^{(3)})\\
&-\tfrac{1}{20}\phi^{(1)}_{,a}\phi^{(1)}_{,b}\phi^{(1)}+\tfrac{4}{105}(\phi^{(3)}\phi^{(3)}_{,b}\phi^{(1)}_{,a}+\phi^{(3)}\phi^{(3)}_{,a}\phi^{(1)}_{,b})-\tfrac{64}{3465}\phi^{(3)}\phi^{(3)}_{,b}\phi^{(3)}_{,a}\\
&-\tfrac{32}{315}(\phi^{(2)}\phi^{(2)}_{,a}\phi^{(3)}_{,b}+\phi^{(2)}\phi^{(2)}_{,b}\phi^{(3)}_{,a})-\tfrac{8}{315}\phi^{(3)}_{,b}\phi^{(3)}_{,a}\phi^{(1)}]
\end{aligned}
$$

$$
\begin{aligned}
N^{(2)}_{a33} = h[&-\tfrac{88}{105}\phi^{(3)}\phi^{(3)}\phi^{(0)}_{,a}+\tfrac{16}{105}\phi^{(2)}_{,a}\phi^{(3)}\phi^{(3)}+\tfrac{8}{35}\phi^{(3)}_{,a}\phi^{(2)}\phi^{(1)}\\
&+\tfrac{8}{105}\phi^{(2)}_{,a}\phi^{(3)}\phi^{(1)}-\tfrac{8}{15}\phi^{(0)}_{,a}\phi^{(3)}\phi^{(1)}-\tfrac{32}{35}\phi^{(2)}\phi^{(3)}\phi^{(1)}_{,a}\\
&-\tfrac{2}{5}\phi^{(2)}\phi^{(1)}_{,a}\phi^{(1)}+\tfrac{32}{105}\phi^{(3)}_{,a}\phi^{(2)}\phi^{(3)}-\tfrac{1}{6}\phi^{(0)}_{,a}\phi^{(1)}\phi^{(1)}\\
&-\tfrac{8}{5}\phi^{(0)}_{,a}\phi^{(2)}\phi^{(2)}+\tfrac{1}{15}\phi^{(2)}_{,a}\phi^{(1)}\phi^{(1)}+\tfrac{16}{35}\phi^{(2)}_{,a}\phi^{(2)}\phi^{(2)}]
\end{aligned}
$$

$$
\begin{aligned}
N^{(2)}_{333} = &-\tfrac{192}{35}\phi^{(2)}\phi^{(2)}\phi^{(3)}-\tfrac{128}{105}\phi^{(3)}\phi^{(3)}\phi^{(3)}-\tfrac{12}{5}\phi^{(1)}\phi^{(2)}\phi^{(2)}\\
&-\tfrac{44}{35}\phi^{(1)}\phi^{(3)}\phi^{(3)}-\tfrac{1}{12}\phi^{(1)}\phi^{(1)}\phi^{(1)}-\tfrac{2}{5}\phi^{(3)}\phi^{(1)}\phi^{(1)}
\end{aligned}
$$

$$
\begin{aligned}
N^{(3)}_{abc} = h^4[&-\tfrac{1}{5}(\phi^{(0)}_{,a}\phi^{(0)}_{,b}\phi^{(1)}_{,c}+\phi^{(0)}_{,a}\phi^{(0)}_{,c}\phi^{(1)}_{,b}+\phi^{(0)}_{,b}\phi^{(0)}_{,c}\phi^{(1)}_{,a})\\
&+\tfrac{4}{35}(\phi^{(3)}_{,c}\phi^{(0)}_{,a}\phi^{(0)}_{,b}+\phi^{(3)}_{,b}\phi^{(0)}_{,a}\phi^{(0)}_{,c}+\phi^{(3)}_{,a}\phi^{(0)}_{,b}\phi^{(0)}_{,c})
\end{aligned}
$$

$$+\tfrac{2}{35}(\phi^{(2)}_{,c}\phi^{(1)}_{,a}\phi^{(0)}_{,b}+\phi^{(2)}_{,b}\phi^{(1)}_{,c}\phi^{(0)}_{,a}+\phi^{(2)}_{,a}\phi^{(1)}_{,c}\phi^{(0)}_{,b}$$

$$+\phi^{(2)}_{,b}\phi^{(1)}_{,a}\phi^{(0)}_{,c}+\phi^{(2)}_{,a}\phi^{(1)}_{,b}\phi^{(0)}_{,c}+\phi^{(2)}_{,c}\phi^{(1)}_{,b}\phi^{(0)}_{,a})$$

$$-\tfrac{1}{28}\phi^{(1)}_{,a}\phi^{(1)}_{,b}\phi^{(1)}_{,c}+\tfrac{1}{63}(\phi^{(3)}_{,c}\phi^{(1)}_{,a}\phi^{(1)}_{,b}+\phi^{(3)}_{,b}\phi^{(1)}_{,a}\phi^{(1)}_{,c}+\phi^{(3)}_{,a}\phi^{(1)}_{,b}\phi^{(1)}_{,c})$$

$$-\tfrac{16}{315}(\phi^{(3)}_{,c}\phi^{(2)}_{,a}\phi^{(0)}_{,b}+\phi^{(3)}_{,b}\phi^{(2)}_{,c}\phi^{(0)}_{,a}+\phi^{(3)}_{,b}\phi^{(2)}_{,a}\phi^{(0)}_{,c}$$

$$+\phi^{(3)}_{,a}\phi^{(2)}_{,b}\phi^{(0)}_{,c}+\phi^{(3)}_{,a}\phi^{(2)}_{,c}\phi^{(0)}_{,b}+\phi^{(3)}_{,c}\phi^{(2)}_{,b}\phi^{(0)}_{,a})$$

$$-\tfrac{8}{315}(\phi^{(2)}_{,c}\phi^{(2)}_{,b}\phi^{(1)}_{,a}+\phi^{(2)}_{,c}\phi^{(2)}_{,a}\phi^{(1)}_{,b}+\phi^{(2)}_{,b}\phi^{(2)}_{,a}\phi^{(1)}_{,c})+\tfrac{32}{3003}\phi^{(3)}_{,c}\phi^{(3)}_{,b}\phi^{(3)}_{,a}$$

$$-\tfrac{8}{693}(\phi^{(3)}_{,c}\phi^{(3)}_{,b}\phi^{(1)}_{,a}+\phi^{(3)}_{,c}\phi^{(3)}_{,a}\phi^{(1)}_{,b}+\phi^{(3)}_{,b}\phi^{(3)}_{,a}\phi^{(1)}_{,c})$$

$$+\tfrac{32}{1155}(\phi^{(2)}_{,c}\phi^{(2)}_{,b}\phi^{(3)}_{,a}+\phi^{(2)}_{,a}\phi^{(2)}_{,b}\phi^{(3)}_{,c}+\phi^{(2)}_{,a}\phi^{(2)}_{,c}\phi^{(3)}_{,b}]$$

$$N^{(3)}_{ab3}=h^3[-\tfrac{64}{3465}(\phi^{(3)}\phi^{(3)}_{,a}\phi^{(2)}_{,b}+\phi^{(3)}\phi^{(3)}_{,b}\phi^{(2)}_{,a})-\tfrac{32}{693}\phi^{(3)}_{,a}\phi^{(3)}_{,b}\phi^{(2)}$$

$$-\tfrac{8}{315}(\phi^{(3)}_{,a}\phi^{(2)}_{,b}\phi^{(1)}+\phi^{(3)}_{,b}\phi^{(2)}_{,a}\phi^{(1)})$$

$$+\tfrac{8}{35}(-\phi^{(3)}\phi^{(0)}_{,a}\phi^{(1)}_{,b}-\phi^{(3)}\phi^{(0)}_{,b}\phi^{(1)}_{,a}+\phi^{(2)}\phi^{(2)}_{,b}\phi^{(0)}_{,a}$$

$$+\phi^{(2)}\phi^{(2)}_{,a}\phi^{(0)}_{,b})-\tfrac{1}{7}\phi^{(2)}\phi^{(1)}_{,a}\phi^{(1)}_{,b}$$

$$+\tfrac{1}{35}(\phi^{(2)}_{,b}\phi^{(1)}_{,a}\phi^{(1)}+\phi^{(2)}_{,a}\phi^{(1)}_{,b}\phi^{(1)})+\tfrac{2}{35}(\phi^{(3)}_{,a}\phi^{(0)}_{,b}\phi^{(1)}+\phi^{(3)}_{,b}\phi^{(0)}_{,a}\phi^{(1)})$$

$$+\tfrac{8}{105}(\phi^{(3)}\phi^{(3)}_{,b}\phi^{(0)}_{,a}+\phi^{(3)}\phi^{(3)}_{,a}\phi^{(0)}_{,b})-\tfrac{32}{315}\phi^{(2)}\phi^{(2)}_{,a}\phi^{(2)}_{,b}$$

$$+\tfrac{4}{105}(\phi^{(3)}\phi^{(2)}_{,b}\phi^{(1)}_{,a}+\phi^{(3)}\phi^{(2)}_{,a}\phi^{(1)}_{,b})+\tfrac{4}{63}(\phi^{(2)}\phi^{(3)}_{,b}\phi^{(1)}_{,a}+\phi^{(2)}\phi^{(3)}_{,a}\phi^{(1)}_{,b})$$

$$-\tfrac{4}{5}\phi^{(2)}\phi^{(0)}_{,a}\phi^{(0)}_{,b}-\tfrac{1}{10}(\phi^{(0)}_{,a}\phi^{(1)}_{,b}\phi^{(1)}+\phi^{(1)}_{,a}\phi^{(0)}_{,b}\phi^{(1)})]$$

$$N^{(3)}_{a33}=h^2[\tfrac{8}{105}\phi^{(3)}_{,a}\phi^{(3)}\phi^{(1)}-\tfrac{1}{20}\phi^{(1)}_{,a}\phi^{(1)}\phi^{(1)}-\tfrac{4}{5}\phi^{(1)}\phi^{(2)}\phi^{(0)}_{,a}$$

$$-\tfrac{8}{35}\phi^{(1)}_{,a}\phi^{(3)}\phi^{(1)}-\tfrac{4}{7}\phi^{(1)}_{,a}\phi^{(2)}\phi^{(2)}$$

$$-\tfrac{64}{35}\phi^{(2)}\phi^{(3)}\phi^{(0)}_{,a}-\tfrac{12}{35}\phi^{(3)}\phi^{(3)}\phi^{(1)}_{,a}+\tfrac{1}{35}\phi^{(3)}_{,a}\phi^{(1)}\phi^{(1)}+\tfrac{8}{35}\phi^{(2)}_{,a}\phi^{(1)}\phi^{(1)}$$

$$+\tfrac{32}{105}\phi^{(2)}_{,a}\phi^{(2)}\phi^{(3)}+\tfrac{16}{165}\phi^{(3)}\phi^{(3)}\phi^{(3)}_{,a}+\tfrac{16}{63}\phi^{(3)}_{,a}\phi^{(2)}\phi^{(2)}]$$

$$N^{(3)}_{333}=h(-\tfrac{16}{7}\phi^{(2)}\phi^{(2)}\phi^{(2)}-\tfrac{3}{5}\phi^{(2)}\phi^{(1)}\phi^{(1)}-\tfrac{144}{35}\phi^{(2)}\phi^{(3)}\phi^{(3)}-\tfrac{96}{35}\phi^{(2)}\phi^{(3)}\phi^{(1)})$$

Appendix II: Transversely isotropic materials

Polarized ceramics are transversely isotropic. Transversely isotropic materials do not constitute a crystal class. Therefore, the matrices for nonlinear material constants of various crystal classes listed by Nelson (1979) cannot be used. Based on the theory of invariants by Spencer (1971), for ceramics poled in the x_3 direction, we can obtain the following constitutive relations cubic in the electric field variables in a similar way as in Yang and Batra (1996).

$$
\begin{Bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \\ \tau_4 \\ \tau_5 \\ \tau_6 \end{Bmatrix}
=
\begin{bmatrix}
c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\
c_{12} & c_{11} & c_{13} & 0 & 0 & 0 \\
c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & c_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & c_{44} & 0 \\
0 & 0 & 0 & 0 & 0 & c_{66}
\end{bmatrix}
\begin{Bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{Bmatrix}
$$

$$
-\begin{bmatrix}
0 & 0 & e_{31} \\
0 & 0 & e_{31} \\
0 & 0 & e_{33} \\
0 & e_{15} & 0 \\
e_{15} & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix}
$$

$$
-\begin{bmatrix}
\hat{b}_{11} & \hat{b}_{12} & \hat{b}_{31} & 0 & 0 & 0 \\
\hat{b}_{12} & \hat{b}_{11} & \hat{b}_{31} & 0 & 0 & 0 \\
\hat{b}_{13} & \hat{b}_{13} & \hat{b}_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & \hat{b}_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & \hat{b}_{44} & 0 \\
0 & 0 & 0 & 0 & 0 & \hat{b}_{66}
\end{bmatrix}
\begin{Bmatrix} (E_1)^2 \\ (E_2)^2 \\ (E_3)^2 \\ 2E_2E_3 \\ 2E_3E_1 \\ 2E_1E_2 \end{Bmatrix}
$$

$$
-E_1\begin{bmatrix}
0 & 0 & 0 & 0 & d_{11} & 0 \\
0 & 0 & 0 & 0 & d_{12} & 0 \\
0 & 0 & 0 & 0 & d_{13} & 0 \\
0 & 0 & 0 & 0 & 0 & d \\
3d & d & d_{44} & 0 & 0 & 0 \\
0 & 0 & 0 & d_{66} & 0 & 0
\end{bmatrix}
\begin{Bmatrix} (E_1)^2 \\ (E_2)^2 \\ (E_3)^2 \\ 2E_2E_3 \\ 2E_3E_1 \\ 2E_1E_2 \end{Bmatrix}
$$

$$
-E_2\begin{bmatrix}
0 & 0 & 0 & d_{12} & 0 & 0 \\
0 & 0 & 0 & d_{11} & 0 & 0 \\
0 & 0 & 0 & d_{13} & 0 & 0 \\
d & 3d & d_{44} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & d \\
0 & 0 & 0 & 0 & d_{66} & 0
\end{bmatrix}
\begin{Bmatrix} (E_1)^2 \\ (E_2)^2 \\ (E_3)^2 \\ 2E_2E_3 \\ 2E_3E_1 \\ 2E_1E_2 \end{Bmatrix}
$$

$$
-E_3\begin{bmatrix} d_{11} & d_{12} & d_{31} & 0 & 0 & 0 \\ d_{12} & d_{11} & d_{31} & 0 & 0 & 0 \\ d_{13} & d_{13} & d_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & d_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & d_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & d_{66} \end{bmatrix}\begin{bmatrix} (E_1)^2 \\ (E_2)^2 \\ (E_3)^2 \\ 2E_2E_3 \\ 2E_3E_1 \\ 2E_1E_2 \end{bmatrix}
$$

$$
\begin{pmatrix} D_1 \\ D_2 \\ D_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & e_{15} & 0 \\ 0 & 0 & 0 & e_{15} & 0 & 0 \\ e_{31} & e_{31} & e_{33} & 0 & 0 & 0 \end{pmatrix}\begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{bmatrix}
$$

$$
+\begin{pmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{11} & 0 \\ 0 & 0 & \varepsilon_{33} \end{pmatrix}\begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix}
$$

$$
+\begin{pmatrix} 0 & 0 & 0 & 0 & \chi_{15} & 0 \\ 0 & 0 & 0 & \chi_{15} & 0 & 0 \\ \chi_{15} & \chi_{15} & \chi_{33} & 0 & 0 & 0 \end{pmatrix}\begin{bmatrix} (E_1)^2 \\ (E_2)^2 \\ (E_3)^2 \\ 2E_2E_3 \\ 2E_3E_1 \\ 2E_1E_2 \end{bmatrix}
$$

$$
+\{\tilde{\chi}_1(E_3)^3+\tilde{\chi}_2E_3[(E_1)^2+(E_2)^2+(E_3)^2]\}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
$$

$$
+\{\tilde{\chi}_2(E_3)^2+\tilde{\chi}_3[(E_1)^2+(E_2)^2+(E_3)^2]\}\begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix}
$$

where

$$
c_{66}=(c_{11}-c_{12})/2,\quad \hat{b}_{66}=(\hat{b}_{11}-\hat{b}_{12})/2,\quad d_{66}=(d_{11}-d_{12})/2
$$

For the linear part of the plate constitutive relations we compute γ_{rs}, ψ_{ks}, ζ_{kj}, $\hat{\zeta}_{kj}$, and $\bar{\zeta}_{kj}$ and obtain

$$[c_{rs}] = \begin{pmatrix} c_{11} & c_{12} & 0 \\ c_{12} & c_{11} & 0 \\ 0 & 0 & c_{66} \end{pmatrix}, \quad [c_{rv}] = \begin{pmatrix} c_{13} & 0 & 0 \\ c_{13} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$[c_{vw}] = \begin{pmatrix} c_{33} & 0 & 0 \\ 0 & c_{44} & 0 \\ 0 & 0 & c_{44} \end{pmatrix}, \quad [c_{vw}^{-1}] = \begin{pmatrix} c_{33}^{-1} & 0 & 0 \\ 0 & c_{44}^{-1} & 0 \\ 0 & 0 & c_{44}^{-1} \end{pmatrix}$$

$$[e_{ks}] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ e_{31} & e_{31} & 0 \end{pmatrix}, \quad [e_{kv}] = \begin{pmatrix} 0 & 0 & e_{15} \\ 0 & e_{15} & 0 \\ e_{33} & 0 & 0 \end{pmatrix}$$

$$[\varepsilon_{kj}] = \begin{pmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{11} & 0 \\ 0 & 0 & \varepsilon_{33} \end{pmatrix}$$

$$[\gamma_{rs}] = \begin{pmatrix} c_{11} - c_{13}^2/c_{33} & c_{12} - c_{13}^2/c_{33} & 0 \\ c_{12} - c_{13}^2/c_{33} & c_{11} - c_{13}^2/c_{33} & 0 \\ 0 & 0 & c_{66} \end{pmatrix}$$

$$[\psi_{ks}] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ e_{31} - e_{33}c_{13}/c_{33} & e_{31} - e_{33}c_{13}/c_{33} & 0 \end{pmatrix}$$

$$[\zeta_{kj}] = \begin{pmatrix} \varepsilon_{11} + e_{15}^2/c_{44} & 0 & 0 \\ 0 & \varepsilon_{11} + e_{15}^2/c_{44} & 0 \\ 0 & 0 & \varepsilon_{33} + e_{33}^2/c_{33} \end{pmatrix}$$

$$[\hat{\zeta}_{kj}] = \begin{pmatrix} 3\varepsilon_{11} + 5e_{15}^2/c_{44} & 0 & 0 \\ 0 & 3\varepsilon_{11} + 5e_{15}^2/c_{44} & 0 \\ 0 & 0 & 3\varepsilon_{33} + 5e_{33}^2/c_{33} \end{pmatrix}$$

$$[\bar{\zeta}_{kj}] = \begin{pmatrix} 5\varepsilon_{11} + 7e_{15}^2/c_{44} & 0 & 0 \\ 0 & 5\varepsilon_{11} + 7e_{15}^2/c_{44} & 0 \\ 0 & 0 & 5\varepsilon_{33} + 7e_{33}^2/c_{33} \end{pmatrix}$$

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